

QUANTUM WEAK TURBULENCE

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Abstract

The study of the phenomenon of quantum weak turbulence is extended by determining the quasiparticle spectrum associated with such a system using a Green's function approach. The quasiparticle spectrum calculated establishes the dissipative regime and the inertial regime, hence a Kolmogorov type of picture.

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1 Introduction

As emphasized by Newell and Zakharov the phenomena of turbulence is not confined simply to fluid motion. Weak turbulence in the general framework is the study of dispersive waves (not necessarily in fluids) with weak non-linear interaction. Such a system can be represented by a Hamiltonian corresponding to the wave equation with a small conservative non linearity. The study

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of weak turbulence is relevant to many physical processes like capillary waves on the surface of water.

It may be noted that the non-linear Schrodinger equation(NLS) $i\Psi_t + \Delta\Psi + \alpha|\Psi|^s\Psi = 0$, a candidate for weak turbulence, is widely used in continuum mechanics, plasma physics and optics. The NLS in the form $i\Psi_t + \Delta\Psi + \alpha|\Psi|^2\Psi = 0, \alpha = \pm 1$ describes a number of physical processes like waves on fluid surface etc. A consequence of weak turbulence is to allow a Kolmogorov like spectra (Kolmogorov-Zakharov spectra) which is exact[1,2]. In this paper the study of weak turbulence in the quantum region described in Ref[3] is continued. With the help of a Green's function method the quasiparticle modes which describe excitations in the system are determined. The quasiparticle spectrum establishes clearly the dissipative regime and hence the range of validity of the approach. In the present study the starting Hamiltonian, in the language of second quantization, takes the form[3]

$$H = \sum_k \omega_k a_k^\dagger a_k + \sum_{k_1 k_2 k_3 k_4} T_{k_1 k_2 k_3 k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \quad (1)$$

The momentum conserving d dimensional δ term has been absorbed in the T matrix. The above Hamiltonian may be identified as the microscopic Hamiltonian for a Bose liquid. The a^\dagger, a are the usual creation and annihilation operators satisfying the commutation relation $[a_k, a_l^\dagger] = \delta_{kl}$. The mode number operator is given by $\hat{n}_k = a_k^\dagger a_k$.

2 Turbulence in a Bose Liquid

We interpret the state in which the occupation numbers are large as a fully developed turbulent system. The reason for this is that the system, as we will show, has a region (inertial region) where there is no dissipation followed by a dissipative region at short distances. In order to study turbulence we replace the a_k, a_k^\dagger by c-numbers to the first approximation. This corresponds to assuming that the system is in a state in which there is large occupation number of different momenta. Such a system could model a driven system not in equilibrium relevant to turbulence. It may be emphasized here that the liquid is not a condensate which exists only at $k = 0$. Such a condensate is postulated for equilibrium calculations. We divide the Hamiltonian into

$H = H_o + H_I$ [3]. Since we are interested in the expectation values of fields between states with large mode numbers it is sensible to include as much of these in the unperturbed Hamiltonian H_o as the calculation permits. We do not need to assume that the coupling to all the quartic terms in the Hamiltonian is small; the diagonal part can be arbitrarily large in our approach. The unperturbed part(H_o) thus contains the quadratic part and the diagonal part of the T matrix i.e terms of the form T_{klkl} . After some simplification we can write

$$H_o = \sum_k \omega_k \hat{n}_k + 2 \sum_{k,l} T_{k,l} \hat{n}_k \hat{n}_l \quad (2)$$

$$H_I = \sum T'_{k_1 k_2 k_3 k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \quad (3)$$

where we have introduced the notation $T_k = T_{kkkk}, T_{kl} = T_{klkl}$ and

$$T'_{k_1 k_2 k_3 k_4} = \begin{cases} T_{k_1 k_2 k_3 k_4} & : k_1 \neq k_3 \text{ or } k_4 \\ 0 & : \text{otherwise} \end{cases} \quad (4)$$

The next step is to calculate $\langle \frac{d\hat{n}_k}{dt} \rangle$, the quantum kinetic equation and look for the solution of the equation $\langle \frac{d\hat{n}_k}{dt} \rangle = 0$ (stationary solution). In order to establish Kolmogorov type solutions for the above equation the following scaling relations have been assumed : $\epsilon(\lambda k) = \lambda^\alpha \epsilon(k)$, $T'(\lambda k_1, \lambda k_2, \lambda k_3, \lambda k_4) = \lambda^\beta T'(k_1, k_2, k_3, k_4)$ and $N(\epsilon) = \epsilon^z$. ϵ_k is the energy corresponding to the mode k and $N(\epsilon)$ is the occupation number of mode having energy ϵ . The solutions corresponding to the stationary state are given by [3]

$$N^{(1)}(\epsilon) = C_1 \epsilon^{\frac{-(\Delta+3)}{3}} \quad (5)$$

$$N^{(2)}(\epsilon) = C_2 \epsilon^{\frac{-(\Delta+4)}{3}} \quad (6)$$

where $\Delta = \frac{3d+2\beta}{\alpha} - 4$. The solutions mentioned above do not correspond to the BE distribution, hence they are the non-equilibrium solutions. We have ignored the two equilibrium solutions to the quantum kinetic equation which also exist. Having shown that such non-equilibrium scaling solutions exist we would like to analyse next the behaviour of the unperturbed Hamiltonian for the Kolmogorov type solutions as stated above. Note the unperturbed Hamiltonian in this approach contains the “diagonal” part of the interaction as described in equation (2).

3 Energy Spectrum of the Unperturbed Hamiltonian (H_o)

We are dealing with a Bose fluid, not in equilibrium, in which the occupation numbers of the modes are very high, i.e. $N_k \gg 1$. Hence we now proceed to replace the operator a_k by $\sqrt{N_k} + b_k$ and a_k^\dagger by $\sqrt{N_k} + b_k^\dagger$ where N_k is a c-number and the operators b_k, b_k^\dagger represent fluctuations around the ground state which have occupation N_k for the k th mode. This ground state will be represented by the ket $|0\rangle$. The above substitution results in the eqn (3) being rewritten as

$$H_o = Const + \left(\sum_k A_k b_k^\dagger b_k + \sum_k B_k b_k^\dagger b_{-k}^\dagger + \sum_k B_k b_k b_{-k} \right) \quad (7)$$

where

$$A_k = 4 \sum_l T_{kl} N_l + \omega_k + 4 T_k N_k \quad (8)$$

$$B_k = 4 T_{k-k} N_k \quad (9)$$

The approximate Hamiltonian, equation (7) constructed in this way can be called the Bogolyubov Hamiltonian. The energy spectrum for the above Hamiltonian may be found out by using the method of Green's functions as briefly described below. We define the Green's function $G^{(1)}_k = \langle 0 | T[b_k^\dagger(t)b_k(0)] | 0 \rangle$ and $G^{(2)}_k = \langle 0 | T[b_{-k}(t)b_k(0)] | 0 \rangle$ where $b_k^\dagger(t)$, $b_{-k}(t)$ are in Heisenberg representation. The time ordered expectation values are taken between the states in which there is macroscopic occupation number. This state is taken to represent the turbulent configuration and will be referred to as the 'ground state' of the system. The time derivatives of $G^{(1)}_k$ and $G^{(2)}_k$ yield,

$$\frac{dG^{(1)}_k}{dt} = -\delta(t) + 2iB_k G^{(2)}_k + iA_k G^{(1)}_k \quad (10)$$

$$\frac{dG^{(2)}_k}{dt} = -2iB_k G^{(2)}_k - iA_k G^{(1)}_k \quad (11)$$

The fourier transform of the above equations with respect to the variable t and the evaluation of $\tilde{G}^{(1)}_k(\omega)$, the fourier transform of $G^{(1)}_k$ leads to

$$\tilde{G}^{(1)}_k(\omega) = -\frac{\omega + A_k}{\omega^2 - (A_k^2 - 4B_k^2)} \quad (12)$$

The poles of $\tilde{G}^{(1)}_k$ are given by $\omega = \pm(A_k^2 - 4B_k^2)^{1/2}$. Hence the energy spectrum of the quasiparticles is given by $E = E_o + \sum_k \epsilon(k)c_k^\dagger c_k$ where

$$\epsilon(k) = \sqrt{(A_k - 2B_k)(A_k + 2B_k)} \quad (13)$$

Here E_o represents the ground state energy and c_k, c_k^\dagger are the annihilation and the creation operators for the quasi-particles. The equation (14) may be solved self-consistently using the expressions (9) and (10) and the scaling form of the following functions: $\omega_{\lambda k} \sim \lambda^2 \omega_k$, $T_{\lambda k} \sim \lambda^\beta T_k$, $N_{\lambda k} \sim \lambda^x N_k$ and $\epsilon(\lambda k) \sim \lambda^\alpha \epsilon(k)$. Let $T_{\lambda k} N_{\lambda k} \sim \lambda^\gamma T_k N_k$ where $\gamma = \beta + x$. If we consider the term $A_k + 2B_k$, under rescaling of the momenta by λ , we have, $A_{\lambda k} + 2B_{\lambda k} = 4\lambda^{\gamma+1} \sum T_{kl} N_l + 4\lambda^\gamma T_k N_k + \lambda^2 \omega_k$. In the infrared limit ($k \rightarrow 0$) and $\lambda \rightarrow 0$, it is seen that the term containing ω_k term is rendered 'irrelevant' for $\gamma < 2$ and for $\gamma > 2$, ω_k is the only relevant term. Hence we have for asymptotically small values of k

$$\epsilon(k) \sim \begin{cases} k^\gamma & : \gamma < 2 \\ k^2 & : \gamma > 2 \end{cases} \quad (14)$$

Similar expressions can be written down for the large k spectrum. Let us bring in the concept of superfluidity here. Consider a body of mass M moving through the fluid and in the process excites a quasiparticle of wave number k and energy $\epsilon(k)$. Conservation of momentum yields,

$$Mv_i = Mv_f + \hbar k \quad (15)$$

For dissipation to occur the following relation amongst the energies should be satisfied,

$$\frac{Mv_i^2}{2} > \frac{Mv_f^2}{2} + \epsilon(k) \quad (16)$$

The above two equations lead to the relation $v_i > \frac{\epsilon(k)}{k}$ where $v_c = \frac{\epsilon(k)}{k}$ is called the critical velocity. If the velocity v of a particle moving through the “fluid” of quasiparticles is less than v_c , where v_c is the critical velocity, then the particle cannot loose energy, the fluid is termed to be a superfluid i.e the external particle moving through this fluid experiences no drag. This is the “turbulent regime” for the system. In this regime there is no viscosity and hence corresponds to the large “Reynolds number” situation. The order of magnitude of v_c is given by $v_c \sim \frac{\epsilon(k)}{k} \sim k^{\alpha-1}$. If $\alpha > 1$, it cannot be termed a superfluid as the critical velocity is zero. For $\alpha \leq 1$ it is a superfluid as the critical velocity is very large for small k values. Let us consider an eddy of size k which is associated with quasiparticle of wavenumber k . As is seen from the expression of v_c the eddy critical velocity is k dependent. The velocity scale of each eddy is set by a different v_c . The fluid analogy is further strengthened by the fact that dissipation occurs when $v > v_c$ i.e for increasing k . We will show that a solution where the energy flow is towards the large k region exists. This is the turbulent flow picture.

Note this solution makes it clear that we are in a non-equilibrium situation. For an equilibrium configuration the system should evolve to it's lowest energy state(i.e small k region).

Considering the total Hamiltonian H of equation (2), the resulting multiscale in the system would manifest in intermittency. It is now evident from equation (15) that $\alpha \leq 1$ corresponds only to the $\gamma < 2$ case from a self-consistent picture. We now equate $\alpha = \gamma$ to find out the α corresponding to the solution given by equations (5) and (6). The solutions are,

$$\alpha = 3d - \beta + 3 \quad (17)$$

$$3d - \beta = -3 \quad (18)$$

As is evident from (18), the solution (6) is unphysical. It has been explained later that the above fact is a consequence of the non-equilibrium nature of the problem. For superfluidity we have $\alpha \leq 1$. Imposing the condition on the equations (16) we obtain the lower bound on β for the K-Z solution which is $\beta > 11$ for $d=3$. The result gives us the condition for superfluidity and also explicitly introduces a cut-off (k_d) beyond which the fluid ceases to be superfluid(turbulent) as the quasiparticle's k dependence follows from that of the interaction term T_k . Since we are dealing with superfluid turbulence here one may like to compare the energy content in the eddy k given by $E_k = \epsilon(k)N_k$ to the Kolmogorov's result in the fluid turbulence, $E_k \sim k^{-\frac{5}{3}}$. It may be emphasized here that we are working in a region that is analogous to the inertial regime in the fluid turbulence and Kolmogorov's law is only valid in the inertial region. The infra red region in our case is defined with respect to the critical velocity of the system.

For consistency of the physical picture we need to study how the energy flows in the model following the techniques due to Zakharov[1]. The quantum

kinetic equation for the mode number per unit energy range, N_ϵ , is given by[3],

$$\begin{aligned} \left\langle \frac{dN_\epsilon}{dt} \right\rangle = & \int d\epsilon_1 d\epsilon_2 d\epsilon_3 [U(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon) (\epsilon_1 \epsilon_2 \epsilon_3 \epsilon)^{-z} (\epsilon_3^z + \epsilon^z - \epsilon_1^z - \epsilon_2^z) \epsilon^{-y} \\ & (\epsilon^y + \epsilon_3^y - \epsilon_2^y - \epsilon_1^y) \delta(\epsilon + \epsilon_3 - \epsilon_2 - \epsilon_1)] \end{aligned} \quad (19)$$

where $y = 3z - p - 3$, $N_\epsilon \sim \epsilon^z$ and $p = \frac{3d+2\beta}{\alpha} - 4$. Under the rescaling of the variables $\epsilon, \epsilon_1, \epsilon_2$ and ϵ_3 by λ we have $U(\lambda\epsilon_1, \lambda\epsilon_2, \lambda\epsilon_3, \lambda\epsilon) = \lambda^p U(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon)$ where $p = \frac{3d+2\beta}{\alpha} - 4$. Denoting the above integral by $K(\epsilon, z, y)$ we have the following relation $K(\epsilon, z, y) = \lambda^{-y-1} K(\lambda\epsilon, z, y)$. Putting $\lambda = \frac{1}{\epsilon}$. Hence we have

$$K(\epsilon, z, y) = \epsilon^{-y-1} K(1, z, y) \quad (20)$$

We now define the current corresponding to the conservation equation for the mode number per unit energy range (N_ϵ) as j . The conservation equation is,

$$\left\langle \frac{\partial N_\epsilon}{\partial t} \right\rangle = \frac{\partial j}{\partial \epsilon} = \epsilon^{-y-1} K(1, z, y) \quad (21)$$

$$\Rightarrow j = -\frac{\epsilon^{-y}}{y} K(1, z, y) \quad (22)$$

j can be identified with the energy current per unit energy range as can be seen from simple dimensional analysis. Considering the stationary solutions i.e. K-Z solutions we have for the case $y = 0$,

$$j = -\frac{\partial K(1, v, y)}{\partial y} \Big|_{y=0} \quad (\text{using L'Hospital's rule}) \quad (23)$$

Considering the other stationary solution i.e $y=1$ we have,

$$j = 0 \quad (24)$$

Since we are considering turbulence, a non-equilibrium phenomenon, only the solution corresponding to $j \neq 0$ is the true solution while $j = 0$ is the unphysical one. Thus the flux analysis rules out the second solution. The fact that α cannot be self-consistently determined from the second solution is manifest in the unphysical nature of the solution as discussed above. Note that the current corresponding to the non-zero solution of equation (21) represents an energy flux from low k end (wavenumber at which energy is injected into the system) to the high k end where dissipation takes place in a manner very similar to one encounters in ordinary fluid turbulence governed by the Navier-Stokes equation (*See Fig 1*).

The results confirm that only one of the scaling solutions found correspond to a non- equilibrium situation in which the energy flux is non-zero.

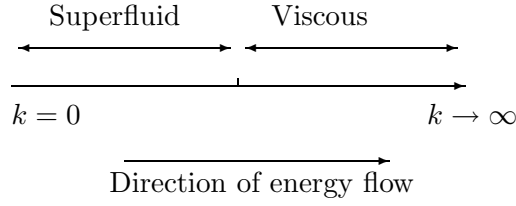


Fig1

In a realistic situation there will be a driving source term for injecting energy into the system. Such systems have been extensively analysed by Zakharov[1]. It is clear that the arguments continue to hold for the general quantum system we have considered. The analysis of the quasiparticle spectrum associated with quantum turbulence thus clearly establishes the inertial range, it places restrictions on parameters, demonstrates that scaling solutions which are not in equilibrium are possible and for the case $\alpha < 1$ i.e where v_c is k dependent suggests that a novel form of intermittency can occur. The analysis thus completes the approach outlined in Ref[3]. We conclude with the following remarks. In the general approach of Zakharov to weak wave turbulence it is shown [1] that non-equilibrium scaling solutions are possible and that there is flux of either momentum or energy present in the system. Our earlier analysis of the quantum version of Zakharov approach [3] established that these features are also present in the quantum system described. In this paper we have gone further. It has been shown that underlying such a non-equilibrium solution there is an inertial range with no dissipation and that a self consistently determined short distance dissipative scale can be found. Thus for the quantum system considered the analysis shows that the original intuition of the Kolmogorov approach to scaling can be justified.

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